Uniform prediction intervals in SAE

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Zurich, 28th August 2018

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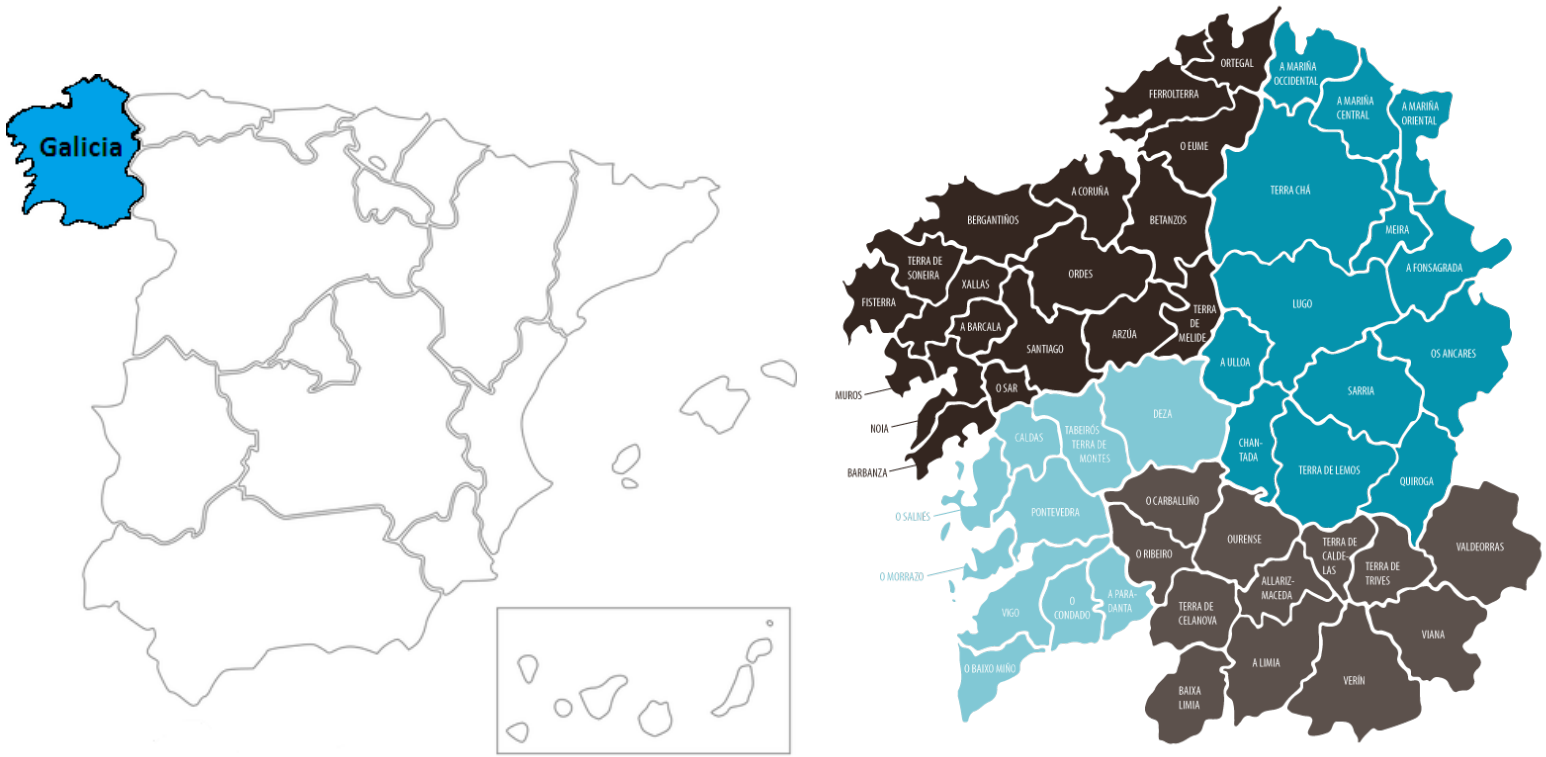
Small Area Estimation

Small **A**rea **E**stimation (SAE) - a family of statistical methods which provides tools for an inference and a prediction for *small areas*, i.e. any entity (a district, a school etc.) for which the direct estimates are not feasible due to the large standard errors (or poor precision).



Countour map of Spain and Galicia.
Source: www.getdrawings.com and www.galiciaenteira.com.

Inference and prediction in SAE

Focus of SAE

- Provides a **reliable estimate** $\hat{\mu}_d$ of the **small area statistic**, $d = 1, ..., D$, i.e. a mean, a count, a ratio, or any other functional if the distribution of area means is estimated.
- Provides a **measure of the variability** of $\hat{\mu}_d$, the most often $\mathrm{MSE}(\hat{\mu}_d)$ and/or **prediction intervals**.
- $\hat{\mu}_d$ is a linear combination (or a function of it) of **fixed** and **random** effects.

Outline

Motivation

SAE allows us to quantify a validity of the policy intervention, health care or environmental projects etc. $\rightarrow$ one **compares simultaneously** small area statistics among all (or some subset of interest) and chooses those which satisfy certain criteria.

Classical pointwise prediction intervals (PPI) at 95% level

1. Comparison of PPIs is **not valid statistically** as we ignore a part of the variability which arises from a simultaneous statement.
2. Coverage probabilities of pointwise prediction intervals are averaged over the areas giving an "approximate picture".
3. **At least** 5% of μ_d are **out of** their **PPI** — effect mainly for areas with **extreme** area effects..
4. The discussion on simultaneous inference in SAE has hardly started, only Ganesh (2009) created **simultaneous Bayesian credible intervals**.

Valid simultaneous inference

Goal

Construction of the tool for simultaneous inference which enables multiple comparisons and provides uniform coverage probability over the areas of interest.

Our tool

MSE based ***S****imultaneous (uniform)* ***P****rediction* ***I****ntervals* (SPI) from frequentist perspective for small area parameters under Liner Mixed Model:

(a) analytical derivation based on the volume of tube formula

(b) Monte Carlo approximation of (a)

(c) bootstrap estimation

Modeling Framework

- Consider LMM: $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{e}$.
- Conditional covariance matrix of $\boldsymbol{y}$: $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{\theta}) = \boldsymbol{R} + \boldsymbol{Z}\boldsymbol{G}\boldsymbol{Z}^t$, $\boldsymbol{\theta}$ variance parameter, e.g. under NERM $\boldsymbol{\theta} = (\sigma_e^2, \sigma_u^2)$.
- BLUE for $\boldsymbol{\beta}$: $\tilde{\boldsymbol{\beta}}(\boldsymbol{\theta}) = (\boldsymbol{X}^t\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^t\boldsymbol{V}^{-1}\boldsymbol{y}$.
- BLUP for $\boldsymbol{u}$: $\tilde{\boldsymbol{u}}(\boldsymbol{\theta}) = \boldsymbol{G}\boldsymbol{Z}\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}})$.
- *Empirical* counterparts $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{u}}$ with $\hat{\boldsymbol{\theta}}$ are estimated using MM, ML, REML, Henderson method etc.

Target parameter: $\bar{Y}_d = N_d^{-1}\sum_{j=1}^{N_d} y_{dj}$

$$\bar{Y}_d \approx \mu_d = \bar{\boldsymbol{X}}_d^t\boldsymbol{\beta} + \bar{\boldsymbol{Z}}_d^t\boldsymbol{u}_d$$
$$\Downarrow$$
EBLUP
$$\Downarrow$$
$$\hat{\mu}_d = \bar{\boldsymbol{X}}_d^t\hat{\boldsymbol{\beta}} + \bar{\boldsymbol{Z}}_d^t\hat{\boldsymbol{u}}_d$$

Outline

General construction of SPI in SAE

Object of interest: **a confidence region** $I_{1-\alpha}$ such that
$P(\mu_d \in I_{1-\alpha} \; \forall d \in \{1, \ldots, D\}) = 1 - \alpha$

- Equivalent with choosing a critical value $c_{1-\alpha}$ such that

$$\begin{aligned} \alpha &= \mathrm{P}\left(|\hat{\mu}_d - \mu_d| > c_{1-\alpha} \cdot \mathrm{MSE}^{1/2}(\hat{\mu}_d) \; \forall d \in \{1, \ldots, D\}\right) \\ &= \mathrm{P}\left(\sup_{1 \leqslant d \leqslant D} \left| \frac{\hat{\mu}_d - \mu_d}{\mathrm{MSE}^{1/2}(\hat{\mu}_d)} \right| > c_{1-\alpha}\right) \end{aligned}$$

- Estimate MSE and approximate quantile $c_{1-\alpha}$ from the distribution of

$$S_D = \sup_{1 \leqslant d \leqslant D} \left| \frac{\hat{\mu}_d - \mu_d}{\mathrm{MSE}^{1/2}(\hat{\mu}_d)} \right|$$

- Construct $I^S_{1-\alpha}$ based on an order statistic of S_D

$$I^S_{1-\alpha} : \left[\hat{\mu}_d - c_{1-\alpha} \times \mathrm{MSE}^{1/2}(\hat{\mu}_d), \hat{\mu}_d + c_{1-\alpha} \times \mathrm{MSE}^{1/2}(\hat{\mu}_d)\right] \; \forall d \in \{1, \ldots, D\}$$

Analytical derivation based on the volume of tube I

Following Sun and Loader (1994), a reasonable formula for the prediction bands for small area means in LMM is

$$\begin{aligned}
\alpha &= \mathrm{P}\left(\sup_{\boldsymbol{x}\in\mathcal{X}} \frac{\left|\boldsymbol{x}^t\boldsymbol{\beta} + \boldsymbol{z}^t\boldsymbol{u} - \hat{\boldsymbol{l}}^t\boldsymbol{y}\right|}{\hat{\sigma}_e||\hat{\boldsymbol{l}}_M||} > c_{1-\alpha} \right) \\
&\leqslant \mathrm{P}\left(\sup_{\boldsymbol{x}\in\mathcal{X}} \left\{ \left[\left| \frac{\langle \boldsymbol{l}_M, \boldsymbol{e}_M \rangle}{\sigma_e||\boldsymbol{l}_M||} \right| + \frac{\eta(\boldsymbol{x})}{\sigma_e||\boldsymbol{l}_M||} \right] \frac{||\boldsymbol{l}_M||}{||\hat{\boldsymbol{l}}_M||} \right\} > c_{1-\alpha}\frac{\hat{\sigma}_e}{\sigma_e} \right) \\
&\leqslant \mathrm{P}\left(\sup_{\boldsymbol{x}\in\mathcal{X}} |\mathcal{Z}| > c_{1-\alpha}\frac{\hat{\sigma}_e}{\sigma_e}\xi - \eta \right) = 2\mathrm{P}\left(\sup_{\boldsymbol{x}\in\mathcal{X}} \mathcal{Z} > c_{1-\alpha}\frac{\hat{\sigma}_e}{\sigma_e}\xi - \eta \right)
\end{aligned}$$

where $\xi = \inf_{\boldsymbol{x}\in\mathcal{X}} \frac{||\hat{\boldsymbol{l}}_M||}{||\boldsymbol{l}_M||}$ and $\eta = \sup_{\boldsymbol{x}\in\mathcal{X}} \frac{\eta(\boldsymbol{x})}{\sigma_e||\boldsymbol{l}_M||}$ with $\eta(\boldsymbol{x})$ a complex remainder.

For consistent $\hat{\boldsymbol{\theta}}$ we have $\xi = 1 + o_p(1)$ and $\eta = o_p(1)$ as $n \to \infty$.

Analytical derivation based on the volume of tube II

Proposition

Suppose that σ_e^2 is estimated by some consistent estimator and that $\exists\ \xi_0 > 0$, $\eta_0 > 0$ such that $\mathrm{P}(\xi \leqslant \xi_0) = o(\alpha)$ and $\mathrm{P}(\eta \geqslant \eta_0) = o(\alpha)$ as $n \to \infty$ and $\alpha \to 0$.
Then, for $\boldsymbol{p = 1}$, α can be approximated by:

$$\alpha \approx \frac{\kappa_0}{\pi}\left[\left(1+\frac{c_{1-\alpha}^2\xi_0^2}{\nu}\right)^{-\nu/2} + \eta_0\frac{2^{1/2}c_{1-\alpha}\xi_0\Gamma\left(\frac{\nu+1}{2}\right)}{\nu^{1/2}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{c_{1-\alpha}^2\xi_0^2}{\nu}\right)^{-(\nu+1)/2}\right]$$
$$+\,\mathcal{E}\cdot P(|t_\upsilon| \geqslant c_{1-\alpha}\xi_0)$$

where t_ν is a t-distributed r.v. with ν d.o.f., $\kappa_0 = \int_{x\in\mathcal{X}} \|\mathcal{Q}'(\boldsymbol{x})\|d\boldsymbol{x}$ with $\mathcal{Q} = \frac{\boldsymbol{l}_M}{\|\boldsymbol{l}_M\|}$ and $\mathcal{E}$ is the Euler-Poincaré characteristic of the manifold $\mathcal{M} = \{\mathcal{Q}(\boldsymbol{x}), \boldsymbol{x} \in \mathcal{X}\}$.

Remarks:

- ξ_0 and η_0 needs to be estimated and geometrical constants — approximated numerically.
- For $p = 2$ and $p \geqslant 3$ the expressions contain more constants to approximate.
- **The volume of tube approximation is not operational and involves substantial numerical and large sample approximation errors**.

Monte Carlo Approximation

Idea

Use Monte Carlo methods to approximate the distribution of the random vector of
$$\hat{\mu}_d - \mu_d = \bar{\boldsymbol{X}}_d^t \left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) + \bar{\boldsymbol{Z}}_d^t \left(\hat{\boldsymbol{u}}_d - \boldsymbol{u}_d\right)$$

- Under the assumption of normality, it follows that
$$\begin{bmatrix} \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \hat{\boldsymbol{u}} - \boldsymbol{u} \end{bmatrix} \approx N\left[\mathbf{0}, \left(\boldsymbol{C}^t \boldsymbol{R}^{-1} \boldsymbol{C} + \boldsymbol{G}^{0+}\right)^{-1}\right]$$
- Define $\bar{\boldsymbol{C}}_d = (\bar{\boldsymbol{X}}_d^t, \bar{\boldsymbol{Z}}_d^t)^t$ and approximate S_D by its simulated counterpart $\hat{S}_D$
$$S_D = \max_{d=1,\ldots,D} \frac{|\hat{\mu}_d - \mu_d|}{\mathrm{MSE}^{1/2}(\hat{\mu}_d)} \approx \max_{d=1,\ldots,D} \frac{\left|\bar{\boldsymbol{C}}_d^t \begin{bmatrix} \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \\ \hat{\boldsymbol{u}}_d - \boldsymbol{u}_d \end{bmatrix}\right|}{\mathrm{MSE}^{1/2}(\hat{\mu}_d)} =: \hat{S}_D$$
- Construct $I_{1-\alpha}^{MC}$ using an order statistic $\hat{c}_{1-\alpha}$ of $\hat{S}_D$

$$I_{1-\alpha}^{MC} : \left[\hat{\mu}_d - \hat{c}_{1-\alpha} \times \mathrm{MSE}^{1/2}(\hat{\mu}_d), \hat{\mu}_d + \hat{c}_{1-\alpha} \times \mathrm{MSE}^{1/2}(\hat{\mu}_d)\right] \quad \forall d \in \{1, \ldots, D\}$$

Bootstrap Approximation

Idea

Use bootstrapping to approximate the distribution of the statistic S

$$S_D^{*(b)} = \max_{d=1,\ldots,D} \left| \frac{\hat{\mu}_d^{*(b)} - \mu_d^{*(b)}}{\mathrm{MSE}^{*1/2}(\hat{\mu}_d)} \right|$$

- Construct $I_{1-\alpha}^B$ based on an order statistic of S_D^*

$$I_{1-\alpha}^B : \left[\hat{\mu}_d - c_{1-\alpha}^* \times \mathrm{MSE}^{1/2}(\hat{\mu}_d), \hat{\mu}_d + c_{1-\alpha}^* \times \mathrm{MSE}^{1/2}(\hat{\mu}_d)\right] \quad \forall d \in \{1, \ldots, D\}$$

- Coverage probability of $I_{1-\alpha}^B$ is asymptotically correct

Theorem

Under necessary regularity conditions it holds that

$$P\left(\mu_d \in I_{1-\alpha}^B \ \forall d \in 1, \ldots, D\right) = 1 - \alpha + O(D^{-1})$$

Outline

Estimation of MSE

Classical decomposition

$$\mathrm{MSE}[\hat{\mu}_d] = \mathrm{MSE}[\tilde{\mu}_d] + \mathrm{E}[\hat{\mu}_d - \tilde{\mu}_d]^2 + 2 \cdot \mathbb{E}\left[(\tilde{\mu}_d - \mu_d)(\hat{\mu}_d - \tilde{\mu}_d)\right]$$

- $\mathrm{MSE}[\tilde{\mu}_d]$ accounts for the variability of μ_d when $\boldsymbol{\theta}$ known; in LMM

$$\mathrm{MSE}[\tilde{\mu}_d] = g_{1d}(\boldsymbol{\theta}) + g_{2d}(\boldsymbol{\theta})$$

  with $g_{1d}(\boldsymbol{\theta})$ and $g_{2d}(\boldsymbol{\theta})$ explicit in $\boldsymbol{X}$, $\boldsymbol{V}$, $\boldsymbol{Z}$, $\boldsymbol{G}$

- Generally, two last terms in above decomposition are intractable.
- We can estimate $\mathrm{MSE}[\hat{\mu}_d]$ using:
  1. Taylor approximation;
  2. different types of bootstrap estimators (approximate the whole expression vs. approximate the intractable terms, correct for bias etc.);
  3. various bootstrap schemes are available (i.e. parametric, wild).

Outline

Simulation settings

- LMM - NERM: $\beta_0 = 1$, $\beta_1 = 1$, $x_{dj} \sim U[0,1]$, $d = 1, \ldots, D$, $j = 1, \ldots, n_d$.
- $n_d = 5, 10, 15$, $D = 25, 50, 75$, different distributions for u_d, e_{dj}.
- $B = 1000$ bootstrap samples with $C = 1$ for double bootstrap.
- 500 simulation runs for evaluating 95% PPI & SPI.
- Results only for random effect bootstrap (REB).

Remarks:

- Since BLUP is unbiased under the assumption of symmetry of F_e and F_u, most methods work only for symmetric distributions
- Strong disturbance if F_e is not normal (CLT); F_u matters less

MC SPI

Empirical coverage probability using $I^{MC}_{1-\alpha}$ and different MSE estimators.
The nominal coverage probability is 95%.

			mse_L		mse^*_{B1}		mse^*_{BC2}		mse^*_{3T}		mse^*_{SP}		mse^*_{SPA}	
e_{dj}	u_d	D/n_d	25/5	75/15	25/5	75/15	25/5	75/15	25/5	75/15	25/5	75/15	25/5	75/15
$t_6(0.5)$	$N(1)$	REML	89.6	92.0	89.4	92.2	89.8	91.8	89.4	92.4	89.6	92.0	89.4	92.0
$t_6(0.5)$	$N(1)$	MM	89.6	92.2	88.8	92.6	88.8	92.6	89.2	92.4	89.6	92.2	89.8	92.2
$\chi^2_5(1)$	$N(0.5)$	REML	89.8	90.4	90.4	90.6	90.8	90.2	90.4	90.6	89.8	90.2	89.8	90.4
$\chi^2_5(1)$	$N(0.5)$	MM	89.6	90.6	90.6	90.4	90.2	90.2	90.2	90.6	89.6	90.6	89.6	90.4
$N(1)$	$N(0.5)$	REML	92.4	94.6	92.2	94.4	91.6	94.6	91.6	94.4	92.4	94.6	92.4	94.6
$N(1)$	$N(0.5)$	MM	92.6	94.8	91.6	94.8	92.0	95.6	92.0	95.0	92.6	94.8	92.6	94.8
$t_6(1)$	$t_6(0.5)$	REML	87.8	91.2	87.8	91.6	88.8	91.8	88.2	91.6	87.8	91.2	87.4	90.8
$t_6(1)$	$t_6(0.5)$	MM	88.2	91.0	88.0	91.8	89.0	92.8	87.8	91.8	88.2	91.0	87.8	91.2

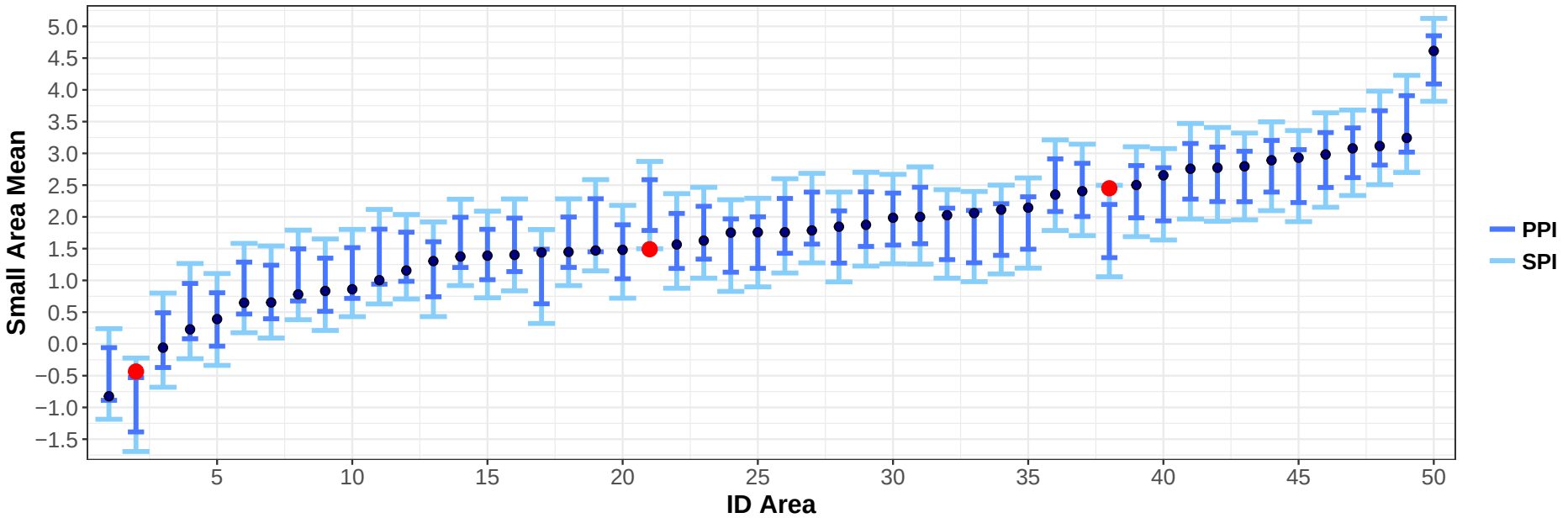
REB SPI

Empirical coverage probability using $I^B_{1-\alpha}$ with REB and different MSE estimators. The nominal coverage probability is 95%.

			mse^*_{B1}		mse^*_{BC2}		mse^*_{3T}		mse^*_{SP}		mse^*_{SPA}	
e_{dj}	u_d	D/n_d	25/5	75/15	25/5	75/15	25/5	75/15	25/5	75/15	25/5	75/15
$t_6(0.5)$	$N(1)$	REML	92.6	95.4	92.2	95.4	92.8	95.6	93.6	95.8	93.6	95.8
$t_6(0.5)$	$N(1)$	MM	92.4	95.4	91.2	95.2	92.6	95.4	93.4	95.6	93.4	95.6
$\chi^2_5(1)$	$N(0.5)$	REML	92.2	93.2	91.8	93.2	92.4	93.4	93.0	93.4	92.6	93.4
$\chi^2_5(1)$	$N(0.5)$	MM	93.0	93.2	92.4	93.0	92.6	93.2	92.8	93.6	93.6	93.6
$N(1)$	$N(0.5)$	REML	92.2	94.2	92.0	94.4	92.4	94.2	92.6	94.6	92.8	94.8
$N(1)$	$N(0.5)$	MM	92.4	95.0	92.6	95.6	92.8	94.6	92.2	95.0	92.4	94.8
$t_6(1)$	$t_6(0.5)$	REML	90.8	94.2	91.0	93.6	91.2	94.0	91.2	95.0	91.2	95.0
$t_6(1)$	$t_6(0.5)$	MM	90.8	95.2	90.8	95.2	90.8	95.2	91.0	95.0	90.6	95.0

Comparison of PPI and REB SPI

PPI and REB bootstrap SPI for small area means estimated using REML, mse^*_{SPA}, $e_{dj} \sim t(0.5)$, $u_d \sim N(1)$, $D = 50$.



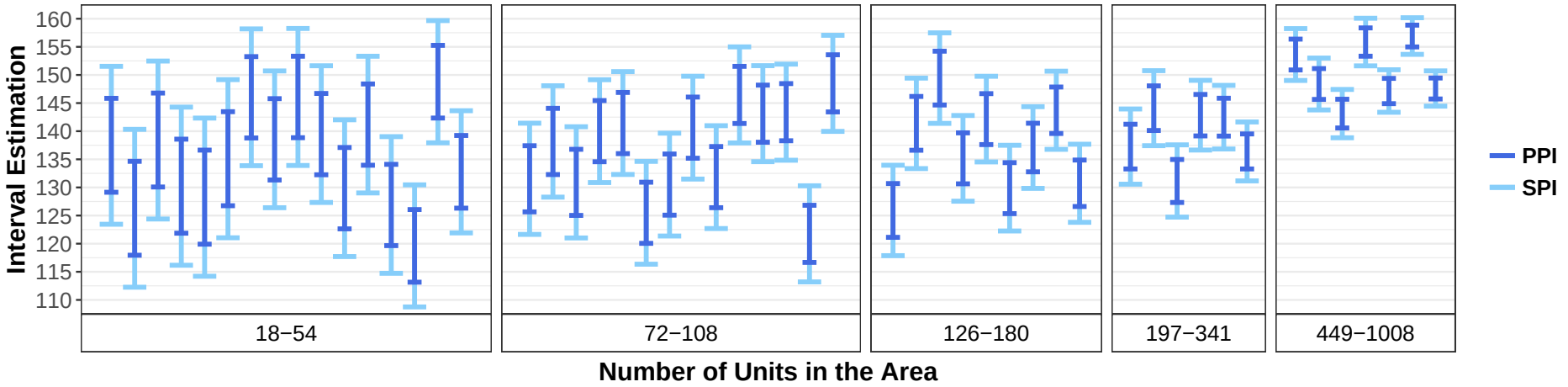
Outline

Modelling income in Galicia: general setting

Structural Survey for Homes in Galicia (NW of Spain)

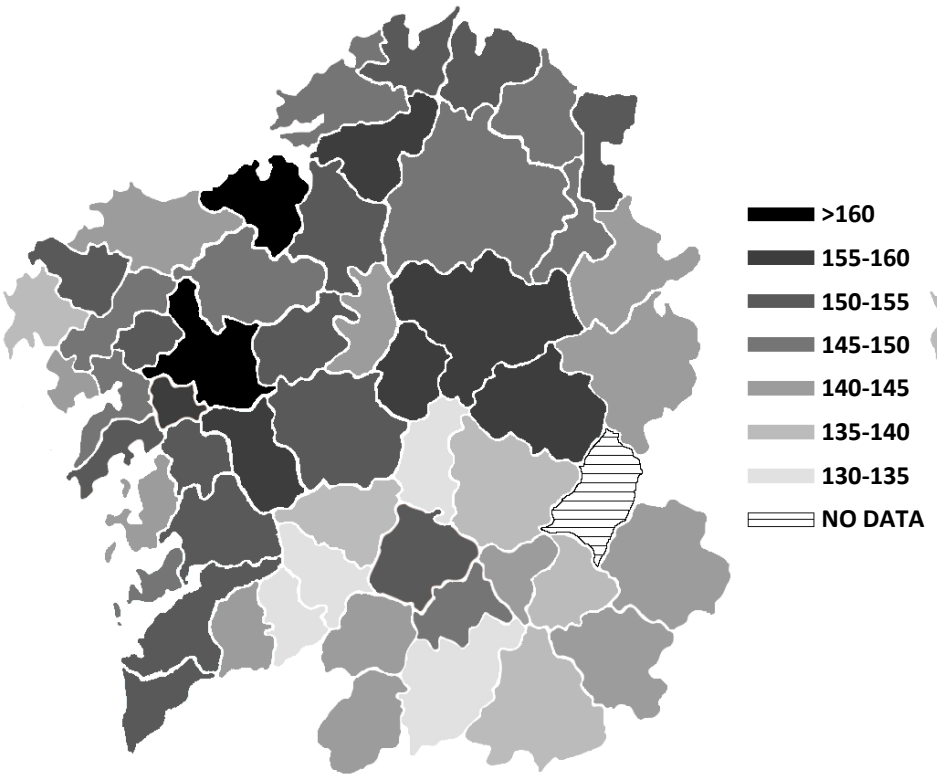
A study of the households and their socio-economical conditions in Galicia in 2005.

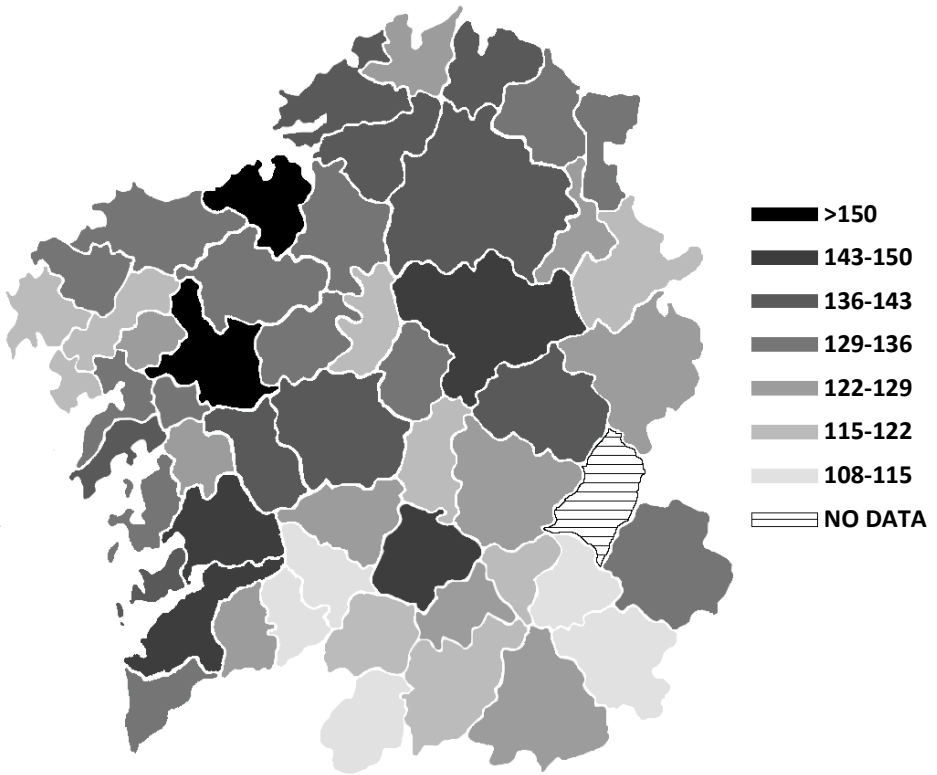
- $\boldsymbol{y}$ is a square root of total income of each household.
- Covariates: selected using xGAIC of Lombardía et al. (2017); $\boldsymbol{x}_1 - \boldsymbol{x}_{11}$ are related to households and $\boldsymbol{x}_{12} - \boldsymbol{x}_{18}$ are related to the person who earns the most in the household.
- Small areas: $D = 52$ comarcas in Galicia with $n_d = 18$ to $n_d = 1008$, $n = 9203$.



Modelling income in Galicia: SPI

SPI estimation of a square root of household income at the level of comarca in Galicia: (left) upper limit, (right) lower limit.





Thank you for the attention.

Outline

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